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STATIONARY PATTERNS FOR REACTION-  
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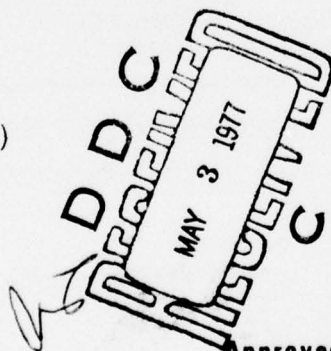
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January 1977

(Received December 12, 1976)



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STATIONARY PATTERNS FOR REACTION-DIFFUSION EQUATIONS

Paul C. Fife

Technical Summary Report #1709  
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ABSTRACT

Patterns are defined to be stable stationary nonconstant solutions of the equations of reaction and diffusion. Several approaches are used to show the existence (or nonexistence) of patterns depending on one variable and defined on the entire real line. For a scalar equation, it is shown that there are essentially no patterns. For a system, small amplitude patterns, larger amplitude "peaks", and larger amplitude "plateaus" are treated. In all cases, stability is an important consideration. Applications to ecology and biophysics are mentioned.

AMS (MOS) Subject Classifications: 35K55, 34C25, 34E15, 35R30, 35B35,  
92A15, 92-00, 92A05

Key Words: Patterns, reaction-diffusion systems, bifurcation of  
periodic solutions, semilinear parabolic systems, stability

Work Unit Number 1 (Applied Analysis)

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Sponsored in part by the United States Army under Contract No. DAAG29-75-C-0024.  
and in part by the National Science Foundation under Grant MPS-74-06835-  
A01. Presented in part at an N.S.F. Regional Conference on Nonlinear  
Diffusion, Houston, June, 1976.

# STATIONARY PATTERNS FOR REACTION-DIFFUSION EQUATIONS

Paul C. Fife

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## I. Introduction

The equations of reaction and diffusion

$$(1) \quad \frac{\partial u}{\partial t} - D\Delta u = f(u), \quad u = (u_1, u_2, \dots, u_n)$$

( $u$  and  $f$  are  $n$ -vectors,  $D$  a matrix, often taken to be diagonal and nonnegative) have been the object of a considerable number of studies in recent years, principally because of their actual and potential applicability to a variety of problems in population dynamics, biophysics, chemical physics, and chemical engineering.

Here we concentrate on the question of existence of stationary "patterns", which we define to be stable, stationary, nonconstant, bounded solutions of (1). (We do not consider "moving patterns" here, such as those studied by Howard and Kopell, Winfree, and others.) Interest in such stationary solutions, sometimes called dissipative structures, was aroused by the work of Turing [27] in the 1950's, and by that of Gmitro and Scriven [15], Prigogine and Nicolis [22] and others in the 1960's and later. Particular interest in them has been occasioned by their possible role in reflecting the corresponding phenomena of

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pattern formation in developing organisms and in ecological communities. However, it should be emphasized that for most contexts in which (1) has been used to model a biological phenomenon (signal dynamics on nerve axons excepted), the model is highly idealized and accounts imperfectly for only part of the processes actually occurring to produce the phenomenon.

Except for remarks at the end of the paper, we restrict attention to solutions depending on only one space variable. This is necessary for some of our results, merely convenient for others. Possible extensions to many-dimensional problems are noted in section IV.

We are particularly interested in patterns which are not elicited by boundary effects, and for this reason further focus attention on patterns defined on the entire real line. Again in the last section, some discussion will be given about corresponding results on bounded domains.

The above definition of pattern requires that it be stable, with reference to the evolution system (1). For some results given here, this means  $C^0$ -stability, which we define as follows: a bounded stationary solution  $\varphi(x)$  of (1) is  $C^0$ -stable if, given any  $\varepsilon > 0$ , there is a  $\delta$  such that every solution  $u(x, t)$  of (1) defined for  $x \in \mathbb{R}$ ,  $t \geq 0$ , satisfying  $|u(\cdot, 0) - \varphi|_0 \equiv \sup_{x \in \mathbb{R}} |u(x, 0) - \varphi(x)| < \delta$  also satisfies  $|u(\cdot, t) - \varphi|_0 < \varepsilon$  for all  $t > 0$ .

In other cases, we are unable to prove  $C^0$ -stability, and instead give stability arguments based on an analysis of the spectrum of the linearization of the right side of (1) about the stationary solution in question. In still other cases, we give less rigorous heuristic arguments favoring stability.

Variations of the above reaction-diffusion problems, such as those obtained by allowing the diffusion terms to be nonlinear, and/or first derivative terms to enter the equation, are important in some applications, but will not be discussed here. Othmer [21] has discussed (among other things) standing oscillation solutions of (1), which bear the same relation to  $x$ -independent oscillatory solutions as do ours to constant solutions.

The paper is concerned with the mathematical analysis of patterns; *experimental and numerical simulation results* are not mentioned except in passing. Nor do we go into much detail about applications in the various fields mentioned above. For an extended discussion of the relevance of patterns in ecology, see (for example) the article by Levin [18].

The plan of the paper is as follows.

- I. Introduction
- II. The scalar case,  $n = 1$
- III. Systems,  $n > 1$ 
  - A. Small amplitude patterns
  - B. Larger amplitude patterns
    - 1. Peaks
    - 2. Plateaus
- IV. Discussion

Throughout, we assume that  $f$  is a continuously differentiable function of its argument.

I gratefully acknowledge helpful discussions with D. H. Sattinger and with K. Kirchgässner, who convinced me that there exist quasiperiodic solutions of (14), and that only the cases shown in Fig. 4 occur.

## II. The scalar case, $n = 1$

In this case,  $D$  is a scalar, which we take to be unity. The equation of one-dimensional patterns, then, is simply

$$(2) \quad u_{xx} + f(u) = 0, \quad x \in \mathbb{R}.$$

This equation is well-known and easily solved by a phase-plane analysis or by the argument below. Furthermore, the stability of its bounded solutions can be tested by means of a technique due to Aronson and Weinberger [1].

The equation has a first integral, obtained by multiplying by the integrating factor  $u_x$ :

$$(3) \quad \frac{1}{2} (u_x)^2 + F(u) = E = \text{const},$$

where

$$F(u) \equiv \int_0^u f(s) ds.$$

If one draws the graph of the "potential"  $V = F(u)$ , as illustrated in Fig. 1, then there is a one-one correspondence between the nonconstant solutions of (2) (modulo translation of the independent variable and reversal of its sign, which are always possible) and horizontal line segments in the  $(u, V)$  plane whose finite endpoints lie on the given curve, and which otherwise lie strictly above the curve. The projection of such a segment onto the  $u$ -axis is the range of the corresponding solution, and the ordinate of the segment is the constant  $E$  (its "energy")



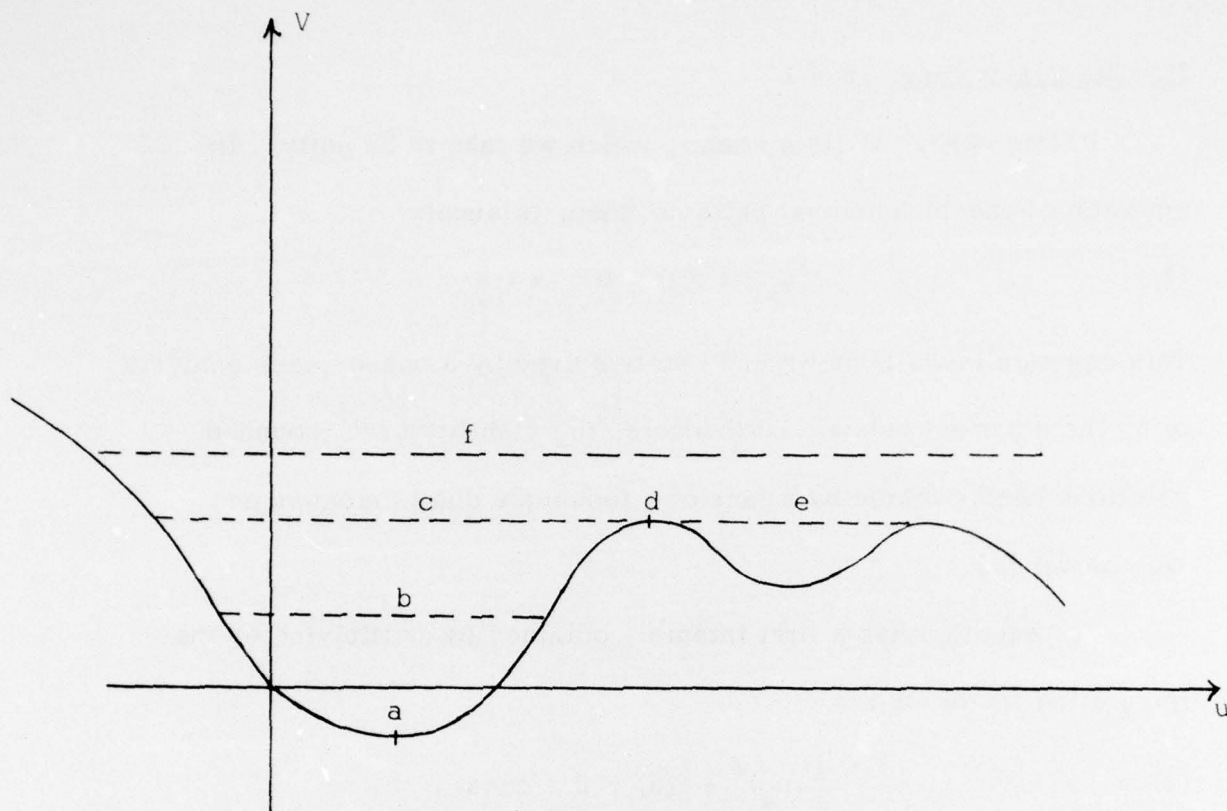


Fig. 1. The potential function  $V = F(u)$ . Horizontal lines (and points) such as  $a - f$  correspond to solutions of (2).

in (3). Given such a segment, the solution  $u(x)$  can be obtained by further integrating (3). On each interval of monotonicity, we obtain

$$(4) \quad x - x_0 = \pm 2^{-\frac{1}{2}} \int_{u_0}^u (E - F(s))^{-\frac{1}{2}} ds ,$$

where  $u_0$  is an arbitrary number in the interior of the (known) range, and  $x_0$  is arbitrary. This excludes the constant solutions of (2); but they, of course, can be identified with the points of zero slope in Fig. 1, since then  $F'(u) = f(u) = 0$ .



Fig. 1 illustrates six solutions: (a) and (d) are constant ones; (b) is a periodic solution; (c) is a solution attaining a minimum at a single finite value of  $x$ , and approaching its supremum as  $x \rightarrow \pm \infty$ ; (e) is a monotone solution approaching different limits as  $x \rightarrow \pm \infty$ ; and (f) is a solution bounded from below, but unbounded above. These properties can be deduced from (4). For example, in cases (c) and (e), the fact that  $|x| \rightarrow \infty$  as  $u$  approaches its supremum  $u_m$  follows from the fact that  $E - F(s) \leq C(u_m - s)^2$  for  $s$  near  $u_m$ , so that the integral in (4) is unbounded as  $u \uparrow u_m$ . In the case (b), a similar analysis shows that the maximum and minimum are obtained for finite values of  $x$ . Then the uniqueness of initial value problems for (2), together with the sign reversibility of  $x$ , show the solution must be periodic.

With all possible solutions of (2) known, we now consider their  $C^0$ -stability as solutions of (1), which in our present case is

$$(5) \quad u_t - u_{xx} - f(u) = 0.$$

The result is that very few are stable.

**Lemma 1:** Every bounded nonconstant solution of (2) which attains a maximum or minimum at a finite value of  $x$  is  $C^0$ -unstable.

**Proof:** Let  $\varphi$  be such a solution. Assume it has a minimum  $\varphi = m$  at  $x = 0$  (the argument for the case of a maximum is the same). Then it corresponds to a segment I, such as that in Fig. 2, whose left

endpoint is on the curve  $V = F(u)$  at a point  $u = m$  such that  $F'(m) < 0$ . Let  $E$  be the ordinate of the segment  $I$  (the energy of  $\varphi$ ), and for positive  $\delta$ , let  $I_\delta$  be the segment with energy  $E + \delta$  overlapping  $I$ . Let  $\varphi_\delta$  be the corresponding (possibly unbounded)

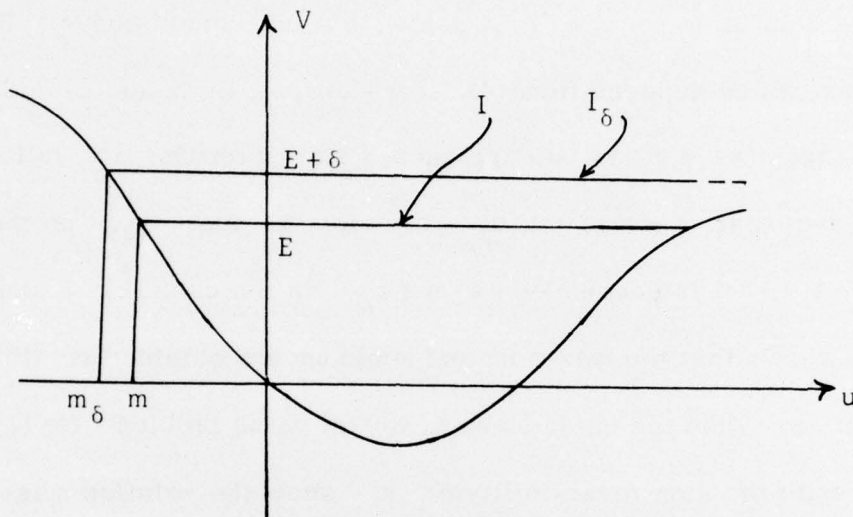


Fig. 2

solution of (2). For small enough  $\delta$ , it will attain a minimum  $m_\delta < m$ , where  $F(m_\delta) = E + \delta$ . Furthermore

$$(6) \quad \lim_{\delta \downarrow 0} m_\delta = m.$$

By translation of the independent variable, if necessary, we may assume that  $\varphi_\delta$  also attains its minimum at  $x = 0$ .

Let  $X_\delta$  be the largest interval on the  $x$ -axis containing the origin, on which  $\varphi_\delta(x) \leq \varphi(x)$ . Since  $I_\delta$  is longer than  $I$  at both

ends,  $\varphi_\delta$  will assume values greater than  $\sup_x \varphi(x)$ . Therefore  $X_\delta$  is finite. Let

$$\bar{\varphi}(x) = \begin{cases} \varphi_\delta(x), & x \in X_\delta; \\ \varphi(x), & x \notin X_\delta. \end{cases}$$

We shall need the following fact, whose proof, being technical, is deferred until later.

Lemma 2:  $\lim_{\delta \downarrow 0} \|\varphi - \bar{\varphi}\|_0 = 0.$

Locally,  $\bar{\varphi}$  is the minimum of two solutions, so will itself be a supersolution of (5). Let  $\bar{u}(x, t)$  be the solution of (5) satisfying  $\bar{u}(x, 0) = \bar{\varphi}(x)$ . By a result of Aronson and Weinberger [1],  $\bar{u}(x, t)$  is decreasing in  $t$ , pointwise in  $x$ . (The proof in [1] was for the case  $\varphi \equiv \text{const}$ , but works as well in our case.)

Let  $[0, T)$  ( $T \leq \infty$ ) be the maximum interval of existence of  $\bar{u}$ . By the decreasing property,  $\lim_{t \uparrow T} \bar{u}(x, t) \equiv \psi(x)$  exists and, where  $\psi > -\infty$ , is a solution of (2) (see [1]).

If  $\psi$  is a solution for all  $x$ , then it must satisfy  $\psi(x) \leq \sup \varphi$  and must attain values  $\leq m_\delta < m$ . This follows from the maximum principle and the fact that  $\bar{\varphi}(x) \leq \varphi(x)$ . But it may be seen that any horizontal segment or point in the  $u - V$  plane corresponding to such a solution must lie entirely to the left of the value  $u = m - \eta$ , for some positive  $\eta$  independent of  $\delta$ . Thus for each  $x$ ,

$$(7) \quad \bar{u}(x, t) \leq m - \eta$$

for large enough  $t$  where  $\bar{u}$  is defined.

On the other hand, if  $\varphi$  is not a solution for all  $x$ , then  $\psi = -\infty$  for some  $x$ , so in any case, (7) holds for at least some values of  $x$  and large enough  $t$ . (It can be shown, in fact, that either  $\psi \equiv -\infty$ , or  $\psi \equiv \text{const}$ , and that  $\bar{u} \rightarrow \psi$  as  $t \rightarrow \infty$  uniformly on bounded sets. But we shall not use these results.)

Now let  $u$  be any solution of (5) satisfying  $u(x, 0) \leq \bar{\varphi}(x)$ . By the maximum principle,  $u(x, t) \leq \bar{u}(x, t)$ , so there are some values of  $x$  for which  $u(x, t) \leq m - \eta$  for all large enough  $t$  where  $u$  is defined. For such  $t$ , we have  $|u(\cdot, t) - \varphi|_0 \geq \eta > 0$ . But in view of Lemma 2, we may choose  $|u(\cdot, 0) - \varphi|_0$  arbitrarily small. Therefore  $\varphi$  is unstable, and the theorem is proved.

Besides the unstable ones covered in the theorem, the only nonconstant bounded solutions of (2) are the monotone ones, corresponding to (e) in Fig. 1. These can be thought of as zero velocity travelling wave solutions in the sense of [13], where their global stability was proved. On the other hand, their existence depends on a special property of the curve  $F$  in Fig. 1, namely that two adjacent local maxima have the same height. A slight change in  $f$  will destroy this property, so the monotone solutions are structurally unstable.

This exhausts the possible bounded solutions of (2), except for the constant ones. Though they are not candidates for being patterns, we include a discussion of their stability for the sake of completeness. As it turns out, a constant solution  $u \equiv u_0$  of (2) is  $C^0$ -stable if



and only if it is stable as a solution of the equation

$$(8) \quad \frac{du}{dt} = f(u) .$$

Suppose first that  $u_0$  is unstable as a solution of (8). Then there exists a neighborhood  $\eta$  of  $u_0$ , and solutions  $u(t)$  of (8) with  $u(0)$  arbitrarily close to  $u_0$ , but with  $u(t) \notin \eta$  for some  $t$ . Such a function  $u(t)$  is also a solution of (5), however. Therefore  $u_0$  is  $C^0$ -unstable as a solution of (5) as well. Next, suppose  $u_0$  is stable as a solution of (8). Then for each  $\varepsilon > 0$ , there is a  $\delta(\varepsilon) > 0$  such that solutions of (8) with  $|u(0) - u_0| \leq \delta$  exist for all time and satisfy  $|u(t) - u_0| < \varepsilon$ . Let  $u(x, t)$  be a solution of (5) with  $|u(\cdot, 0) - u_0| \leq \delta$ . By the maximum principle, it is bounded pointwise above and below, respectively, by  $u_+(t)$  and  $u_-(t)$ , which are solutions of (8) assuming initial values  $u_{\pm}(0) = u_0 \pm \delta$ . Therefore  $|u(\cdot, t) - u_0| < \varepsilon$ , and  $u(x, t)$  is stable.

Altogether, we have the following theorem.

Theorem: The only stable solutions of (2) are the strictly monotone ones, and the constant ones which are stable as solutions of (8). Moreover, the strictly monotone ones are structurally unstable.

Proof of Lemma 2: First, consider the case when  $\varphi$  is periodic. Then geometrical considerations show that the length of  $X_\delta$  will not surpass twice the wave-length of  $\varphi$ , so remains bounded independently of  $\delta$ .

Then the continuity of solutions of initial value problems with respect

to initial data easily implies  $\lim_{\delta \downarrow 0} |\bar{\varphi}(x) - \varphi(x)| = \lim_{\delta \downarrow 0} (\varphi(x) - \varphi_\delta(x)) = 0$ ,



uniformly in  $X_\delta$ . But outside  $X_\delta$ ,  $\bar{\varphi} \equiv \varphi$ , so the conclusion follows.

The only other case is when  $\varphi$  approaches a limit asymptotically as  $|x| \rightarrow \infty$ , and  $\varphi' > 0$  for  $x > 0$ . We treat this case as follows. By assumption,  $F'(m) = f(m) < 0$ . Let  $\kappa = \frac{1}{2} f(m) < 0$ , and restrict  $\delta$  to be so small that  $f(u) < \kappa$  for  $m_\delta \leq u \leq m$ . Then

$$\begin{aligned}\varphi_\delta(x) &= \varphi_\delta(0) + \int_0^x \int_0^s \varphi_\delta''(\xi) d\xi ds \\ &= \varphi_\delta(0) - \int_0^x \int_0^s f(\varphi_\delta(\xi)) d\xi ds \\ &\geq m_\delta + \frac{1}{2} |\kappa| x^2,\end{aligned}$$

for  $x$  such that  $m_\delta \leq \varphi_\delta(x) \leq m$ . Let  $\omega_\delta^2 = 2(m - m_\delta)/|\kappa|$ , so that  $m_\delta + \frac{1}{2} |\kappa| \omega_\delta^2 = m$ . Then  $\varphi_\delta(\omega_\delta) \geq m$ . If we define  $\pi_\delta$  as the first value of  $x > 0$  at which  $\varphi_\delta(x) = m$ , then  $\pi_\delta \leq \omega_\delta$ . It follows that

$$(9) \quad \lim_{\delta \downarrow 0} \pi_\delta = 0.$$

Let  $\psi_\delta(x) \equiv \varphi_\delta(x + \pi_\delta)$ . Then  $\psi_\delta(0) = m = \varphi(0)$ , and from (3), for  $x > 0$ ,  $\psi_\delta' = [2(E + \delta - F(\psi_\delta))]^{\frac{1}{2}}$ , whereas  $\varphi' = [2(E - F(\varphi))]^{\frac{1}{2}} < [2(E + \delta - F(\varphi))]^{\frac{1}{2}}$ .

It follows that  $\psi_\delta(x) \geq \varphi(x)$ , for  $x > 0$ .

We now have, for  $x \geq \pi_\delta$ ,

$$\begin{aligned}\varphi(x) - \varphi_\delta(x) &= \varphi(x) - \varphi(x - \pi_\delta) + \varphi(x - \pi_\delta) - \psi_\delta(x - \pi_\delta) \\ &\leq \varphi(x) - \varphi(x - \pi_\delta).\end{aligned}$$

By the uniform continuity of  $\varphi$  and (9), we have that  $\varphi(x) - \varphi_\delta(x) \leq \sigma_\delta \downarrow 0$

as  $\delta \rightarrow 0$ , for  $x \geq \pi_\delta$ . By symmetry, the same is true for  $x \leq -\pi_\delta$  as well, and by the continuity of  $\varphi$  and  $\varphi_\delta$ , we may extend the relation to the interior interval  $[-\pi_\delta, \pi_\delta]$ .

On  $X_\delta$  we therefore have

$$(10) \quad |\bar{\varphi}(x) - \varphi(x)| = \varphi(x) - \varphi_\delta(x) \leq \sigma_\delta,$$

whereas outside  $X_\delta$ , we have  $|\bar{\varphi}(x) - \varphi(x)| = 0$ . This completes the proof.

### III. Systems ( $n > 1$ )

#### A. Small amplitude perturbations.

Let us consider systems (I) in which  $f(0) = 0$ , so that  $u \equiv 0$  is a solution. It has been observed in the past that this trivial solution may be unstable with respect to certain perturbations which are not constant in  $x$ , while remaining stable with respect to constant (uniform) perturbations. Another way of saying this is that the trivial solution  $u \equiv 0$  may be unstable, whereas the point  $u = 0$  is a stable critical point of the associated kinetic equations

$$(II) \quad \frac{du}{dt} = f(u) .$$

In such cases, the effect of introducing diffusion and  $x$ -dependence is therefore to destabilize the system, somewhat contrary to one's intuition of diffusion as a stabilizing influence. An argument was advanced by Segel and Jackson [23] to partially explain this paradox. If one can characterize some of the components of  $u$  as "stabilizing" in some sense, and others as "destabilizing", and if the stabilizers diffuse more rapidly than the destabilizers, then a very small initial nonuniform concentration of both (this being the perturbation of the trivial solution) may result in the stabilizers diffusing away, their effect thereby diminishing. This would happen to some extent not only with the initial concentration of stabilizers, but also with stabilizers produced later through the reaction process. In this way, without as much counteracting

influence, the destabilizers could take over, and the solution could grow to finite size. For this to happen, it is of course necessary that the diffusion coefficients not all be the same. In the example in [24],  $n = 2$  and the system represented predator-prey equations, the predators being stabilizers and the prey destabilizers. Segel and Levin [25] have pursued similar problems and have shown that after instability sets in, one can expect solutions of (1) with small initial data sometimes to evolve into patterned solutions.

We consider the system (1), and assume the function  $f$  depends on a real parameter:  $f = f(u, \lambda)$ . We also assume that there is a critical number  $\lambda_{cr}$  such that for  $\lambda < \lambda_{cr}$ , the trivial solution of (1) is stable, in a sense to be explained below, whereas for  $\lambda > \lambda_{cr}$ , it is not. The new instability is respect to periodic perturbations with wave length in a certain range, bounded above and below (the meaning of this is given below). In particular, the zero solution remains stable to constant perturbations, so the instability is of the type indicated in the previous paragraph. Under certain additional conditions, the effect of this is that periodic patterns of small amplitude arise for  $\lambda$  in a right-hand neighborhood of  $\lambda_{cr}$ . In fact, we shall show how bifurcation theory may be used to construct a two-parameter family (the parameters being amplitude and wave-length) in a neighborhood of  $\lambda_{cr}$ . The question of their stability will be taken up later.



Our assumptions need to be made precise. The source term  $f$  will be assumed twice continuously differentiable for  $u$  near the origin; we write  $A_\lambda$  for the Jacobian  $\frac{\partial f}{\partial u}(0, \lambda)$ . For simplicity, we assume the dependence of  $A_\lambda$  on  $\lambda$  is linear:  $A_\lambda = A + \lambda B$ , for some matrices  $A$  and  $B$ . The linear problem associated with (1),

$$V_t = DV_{xx} + AV + \lambda BV,$$

has solutions of the form  $V = \Phi \exp[ikx + \sigma t]$  for arbitrary real wave number  $k$ . Here  $\Phi$  will be an eigenvector of the matrix

$$H(\lambda, p) \equiv -pD + A + \lambda B \quad (p = k^2),$$

with corresponding eigenvalue  $\sigma$ .

Assumption 1: There is a number  $\lambda_{cr}$  and a number  $\delta > 0$ , such that for  $|\lambda - \lambda_{cr}| < \delta$  and for all  $p \geq 0$ ,  $H(\lambda, p)$  has a unique eigenvalue with largest real part, and it is real and simple. Denote it by  $\sigma_1(\lambda, p)$ , and let  $I_\lambda = \{p : \sigma_1(\lambda, p) \geq 0\}$ . Then

$$I_\lambda = \begin{cases} \emptyset & \text{for } \lambda_{cr} - \delta < \lambda < \lambda_{cr}, \\ \{p_0\} & \text{for } \lambda = \lambda_{cr}, \text{ where } p_0 > 0, \\ \text{a bounded interval of positive length} & \text{for } \lambda_{cr} < \lambda < \lambda_{cr} + \delta. \end{cases}$$

This assumption about  $\sigma_1(\lambda, p)$  is depicted in Fig. 3.

For simplicity we take  $\lambda_{cr} = 0$  from now on (except in the example below).

It follows from Assumption 1 that the matrix  $H_0 = H(0, p_0)$  has zero as a simple eigenvalue. Its nullspace is therefore spanned by some



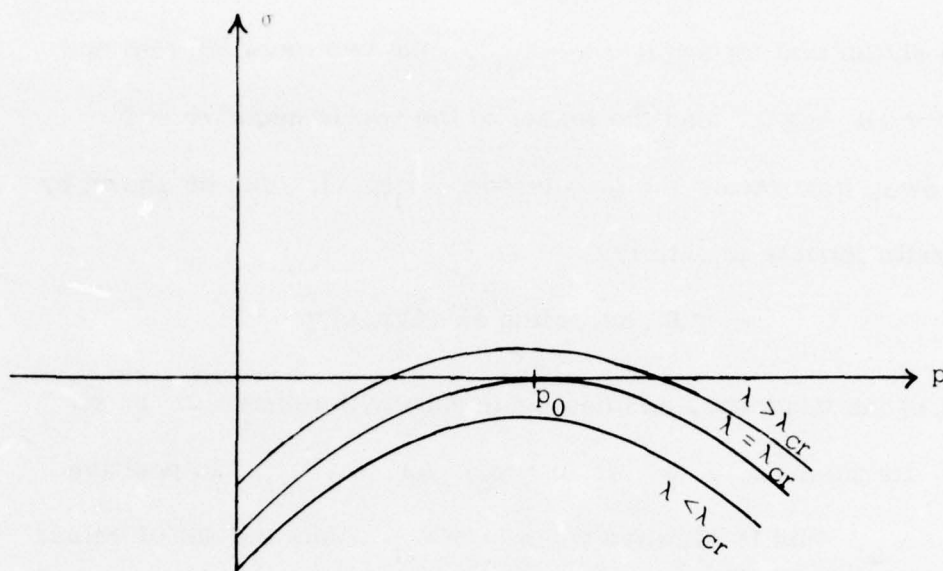


Fig. 3. Typical dependence of the largest eigenvalue  $\sigma_1$  on  $\lambda$  and  $p$ , when Assumption 1 is valid. The curves are for  $\lambda$  constant.

real nullvector  $\Phi$ , and the nullspace of the adjoint matrix is spanned by some real vector  $\Psi$ .

Assumption 2:  $\Psi B \Phi \neq 0$ .

Example: Let  $n = 2$ ,

$$A = \begin{bmatrix} 0 & -2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then Assumption 1 is satisfied with  $p_0 = \sqrt{2} - \frac{1}{2}$ ,  $\lambda_{cr} = \frac{1}{2}(4\sqrt{2} - 1)$ .

In fact, the eigenvalues  $\sigma$  of  $H(\lambda, p)$  are the roots of the equation  $\sigma^2 + \sigma(3p + 1 - \lambda) + Q(p, \lambda) = 0$ , where  $Q(p, \lambda) = (p - \lambda)(2p + 1) + 4$ .

It can be shown that for small  $|\lambda - \lambda_{cr}|$ , the two roots are real and distinct for all  $p \geq 0$ , and the lesser of the two is negative and bounded away from zero. The greater one,  $\sigma_1(p, \lambda)$ , can be shown by the quadratic formula to satisfy

$$\sigma_1 \begin{matrix} \geq \\ < \end{matrix} 0 \text{ according as } Q(p, \lambda) \begin{matrix} \leq \\ > \end{matrix} 0.$$

But  $Q(p, \lambda)$  is quadratic, and has a minimum with respect to  $p$  for each  $\lambda$ . Its minimum is 0 (at  $p = p_0$ ) for  $\lambda = \lambda_{cr}$ , is positive when  $\lambda < \lambda_{cr}$ , and is negative when  $\lambda > \lambda_{cr}$ . Thus the set of values of  $p$  for which  $\sigma_1 > 0$  is empty for  $\lambda < \lambda_{cr}$ , and is a finite interval for  $\lambda > \lambda_{cr}$ .

The basic result is the following.

Theorem: Assume

$$(12) \quad f(u, \lambda) = (A + \lambda B)u + G(u),$$

where  $G$  is continuously differentiable near  $u = 0$ , and

$$(13) \quad |G(u)| \leq C|u|^2$$

for  $|u| < 1$ . Let  $A$ ,  $B$ , and  $D$  satisfy Assumptions 1 and 2. Let

$k_0 = \sqrt{p_0}$  be the "initial" wave number. Then there exists a two-

parameter family of periodic functions  $u(x; \epsilon, \kappa)$ , and a scalar  $\lambda(\epsilon, \kappa)$ ,

both continuous in  $\epsilon$  and  $\kappa$  and defined for small  $|\epsilon|$ ,  $|\kappa|$ , such that

(i)  $u$  and  $\lambda$  satisfy

$$(14) \quad D u_{xx} + (A + \lambda B)u + G(u) = 0, \quad x \in \mathbb{R},$$

for each fixed  $\epsilon$  and  $\kappa$ ,

(ii)  $u$  is periodic in  $x$  with wave-length (period)  $2\pi/k$

where  $k = k_0 + \kappa$ ,

$$(iii) \int_0^{2\pi/k} u(x; \varepsilon, \kappa) \cdot \varphi \cos kx dx = \varepsilon,$$

$$(iv) \lambda(0, 0) = 0, u(x; 0, 0) \equiv 0.$$

We shall sketch the proof. The first step will be to give the problem a more convenient formation. By using the new variable  $\bar{x} = x/k$ , we normalize the wave-length of our solutions to be  $2\pi$ , but at the same time introduce an extra parameter  $k^2 = p$  into the equations (14), which become

$$(15) \quad p \frac{Du}{dx^2} + (A + \lambda B)u + G(u) = 0.$$

Henceforth we shall omit the bars on the  $x$ 's. For  $p = p_0$ ,  $\lambda = 0$ , the corresponding linear equation

$$p_0 \frac{DU}{dx^2} + AU = 0$$

has exactly two linearly independent solutions bounded for all  $x$ :  $\varphi = \varphi \cos x$  and  $\varphi' = -\varphi \sin x$ . This follows from Assumption I. We normalize our

solutions so that  $(\varphi, \varphi) \equiv \int_0^{2\pi} |\varphi|^2 dx = |\varphi|^2 \pi = 1$ . Similarly, the

linear adjoint problem has two linearly independent solutions  $\psi = \Psi \cos x$  and  $\psi' = -\Psi \sin x$ .

Let  $P_1, P_2$  be  $\mathcal{L}_2$ -orthogonal projections onto  $\text{span } \{\psi\}$  and  $\text{span } \{\psi'\}$  respectively; for example,  $P_1 f = (f, \psi)\psi$ . Also let  $Q = \text{id} - P_1 - P_2$ .

Let  $Z$  be the Banach space of continuous  $2\pi$ -periodic functions with values in  $\tilde{R}^n$ ,  $C^k$  the space of  $\tilde{R}^n$ -valued functions with bounded continuous derivatives of order  $k$ ,  $X = Z \cap C^2$  and  $Y$  the  $\mathfrak{L}_2$  orthocomplement of  $\varphi$  and  $\varphi'$  in  $X$ . We norm  $X$  and  $Y$  by the usual  $C^2$  norm, and  $Z$  by the  $C^0$  norm.

Then it is standard that the differential operator

$$L = p_0 D \frac{d^2}{dx^2} + A,$$

restricted to  $Y$ , is one-one from  $Y$  onto  $QZ$ , so that  $L^{-1}$  exists as a bounded operator from  $QZ$  to  $Y$ .

Since  $X = \text{span}\{\varphi, \varphi'\} \oplus Y$ , any solution in  $X$  can be written as  $u = \alpha\varphi + \beta\varphi' + v = \Phi(\alpha \cos x + \beta \sin x) + v$ ,  $v \in Y$ . Furthermore since (15) is autonomous, any solution remains a solution when subjected to an arbitrary translation of the independent variable. In fact, such a translation, does not remove  $v$  from  $Y$ . We suppose our solutions have all been so translated in such a way that  $\beta = 0$ ; we then let  $\varepsilon = \alpha$  and call it the "amplitude" of the solution. In short, we seek solutions in the form

$$(16) \quad u = \varepsilon(\varphi + w), \quad w \in Y.$$

With the two final definitions  $F(u, \varepsilon) = \varepsilon^{-1}G(\varepsilon u)$  (recall (13)) and  $q = p - p_0$ , we are ready for the reformulation. Setting (16) into (15) and using the above definitions, we see that the problem is to find a function  $w \in Y$  satisfying



$$(17) \quad Lw + \lambda B(\varphi + w) + qD(\varphi + w)'' + \varepsilon F(\varphi + w, \varepsilon) = 0 .$$

We rewrite this as  $Lw + R(w; \lambda, \varepsilon, q) = 0$ , where for each  $(\lambda, \varepsilon, q)$ ,  $R$  is an operator from  $Y$  into  $Z$ . Applying projections  $Q$ ,  $P_1$ , and  $P_2$ , we see that (17) is equivalent to the three equations

$$(18a) \quad Lw + QR(w; \lambda, \varepsilon, q) = 0 ,$$

$$(18b) \quad P_1 R(w; \lambda, \varepsilon, q) = 0 ,$$

$$(18c) \quad P_2 R(w; \lambda, \varepsilon, q) = 0 .$$

With the aid of Assumption 2 and the implicit function theorem, the first two equations (18a, b) may be solved uniquely for  $w(\varepsilon, q)$  and  $\lambda(\varepsilon, q)$  for small  $(\varepsilon, q)$ . It remains to be shown that this solution will automatically satisfy (18c). Here we use the following simple invariance property of our problem. The operators  $L$ ,  $R$ ,  $Q$ , and  $L^{-1}Q$  preserve the class of even functions. This fact is immediate for  $L$ ,  $R$ , and  $Q$ , and not too difficult for  $L^{-1}Q$ . It follows that (18a-b) may also be solved in the class of even functions, so by uniqueness, the original solution must be even. But  $P_2$ , being orthogonal projection onto a space of odd functions, annihilates even functions. Therefore (18c) is satisfied, completing the proof of the theorem.

Equations (18) lend themselves to the calculation of explicit approximations to  $w$  and  $\lambda$  as power series in  $\varepsilon$  and  $q$ . Upon calculating these power series to terms of order two, one finds

$$(19) \quad \lambda = \gamma_1 \varepsilon^2 + \gamma_2 \varepsilon q + \gamma_3 q^2 + \text{higher order terms} ,$$



where the coefficients  $\gamma_i$  can be written explicitly in terms of various combinations of the operators  $B, D \frac{d^2}{dx^2}, L^{-1}Q, P_1, F(\cdot, 0)$ , and first order differentials of  $F$ , acting on the function  $\varphi$ . The fact that there are no first order terms in the expression for  $\lambda$  is a consequence of the facts that  $F(\cdot, 0)$  is quadratic or zero, that  $\varphi$  is a vector multiple of  $\cos x$ , and that  $\frac{\partial \sigma_1}{\partial p}(0, p_0) = 0$  (in terms of Fig. 3,  $\sigma_1(\lambda_{cr}, p)$  has a maximum at  $p = p_0$ ).

If we further assume that  $\frac{\partial^2 \sigma_1}{\partial p^2}(0, p_0) < 0$  and  $\frac{\partial \sigma_1}{\partial \lambda}(0, p_0) > 0$  (which inequalities are reasonable, from Assumption 1), then it follows that  $\gamma_3 > 0$ . In this case, it can be seen that the "bifurcation diagram" is generically of one or the other of the two types shown in Fig. 4. In that figure, the shaded regions consist of those pairs  $(\lambda, \varepsilon)$  for which a periodic solution  $u = \varepsilon(\varphi + w)$  exists. These are local diagrams:  $\varepsilon, q$ , and  $\lambda$  are all assumed small.

In the context of problem (14), we have shown that when increasing the value of  $\lambda$  causes the trivial solution to lose its stability, it also causes many new periodic solutions to appear. There are two important questions now to be asked: (i) are there any other small bounded solutions besides the periodic ones we have constructed; and (ii) which of our new solutions, if any, are stable? These are difficult questions, and no complete answer to them exists at this time. However, some comments can be made.

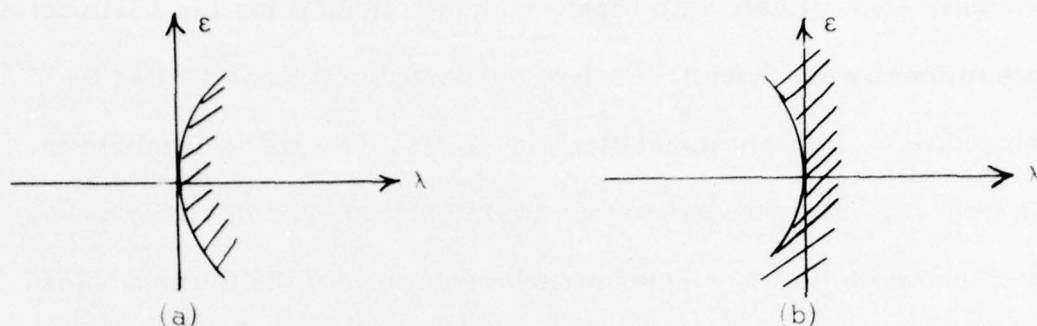


Fig. 4. The shaded areas are the totality of small  $(\lambda, \varepsilon)$  for which a periodic solution exists, with wave number near  $k_0$ :  
 (a) the case  $\gamma_4 \equiv \gamma_1 - \gamma_2^2(4\gamma_3)^{-1} > 0$ ; (b) the case  $\gamma_4 < 0$ .

Regarding the first question, we have constructed all possible periodic solutions with wave number near the one  $(k_0)$  with respect to which instability first sets in. This is clear by the implicit function theorem's uniqueness assertion. However, there undoubtedly exist quasiperiodic solutions as well, as Kirchgässner has argued in his treatment of another type of model problem [17]. In fact, this is clearly true of (17) in the case  $\varepsilon = 0$  (the limit equation for small amplitude solutions). This equation is linear with constant coefficients, and for each small  $\lambda > 0$ , there are exactly two values of  $q$  for which it has even periodic solutions. In terms of the original formulation (14), they correspond to two infinitesimally small even periodic solutions, with distinct periods. Such solutions of the linear problem may be superposed; linear combinations of them yield new quasiperiodic solutions (which are

actually also periodic with possibly large period, if the two basic periods are rationally dependent). Each of the basic solutions may also be subjected to arbitrary translations in  $x$ , so they are no longer even. In this way, two more parameters are introduced, and in all we have a four-parameter family of quasiperiodic solutions of the linear problem. For fixed  $\lambda > 0$ , one expects quasiperiodic solutions to exist also for  $\varepsilon > 0$ , and there are ways to construct formal approximations to them in some cases. If it could be shown that a four parameter family of them exists, for some fixed  $\lambda > 0$ , then it would follow from a result on manifolds of neutral stability [8] that in fact there can be no other small bounded solutions besides those, and the answer to (i) would be more nearly complete.

As for the second question, concerning stability, no completely rigorous results will probably be obtained for some time. Nevertheless, heuristic results on linearized stability may be within reach. Specifically, on linearizing the operator on the left of (17) about a given periodic solution  $u_{\varepsilon, q} = \varepsilon(\varphi + w_{\varepsilon, q})$ , one obtains an operator with periodic coefficients, depending also on the parameters  $\varepsilon$  and  $q$ . One then would look for all eigenvalues of this operator, acting on  $C^2$  functions, and ask for a relation between  $\varepsilon$  and  $q$  which will guarantee that no such eigenvalues lie in the right half-plane ( $0$  is always an eigenvalue). Such pairs  $(\varepsilon, q)$  would then yield periodic solutions which may be stable in some sense.

By a Floquet-type argument, it is sufficient to pose this eigenvalue problem in the class of functions of the form  $e^{i\alpha x}v(x)$  with  $v$   $2\pi$ -periodic and  $\alpha$  real. Then the set of eigenvalues will depend on  $\alpha$  as well as on  $\varepsilon$  and  $q$ , and one can restrict attention to small  $\alpha$ , because otherwise all eigenvalues can be shown to have negative real part.

My conjecture is that in case (a) of Fig. 4, only the solutions with extremal  $\varepsilon$  are stable in the above sense, whereas in case (b), none are stable except the trivial solution with  $\lambda < 0$ .



### III. B. 1. Larger amplitude patterns: peaks

There has been a sizeable amount of numerical simulation in recent years directed toward the discovery of finite amplitude patterns in model reaction-diffusion systems. The major efforts have been by a group in Brussels (see, for example, [3], [16]), and by Gierer and Meinhardt ([14], [19], [20]) in Tübingen. These efforts have indeed been successful, as one can judge by reading the cited papers.

The principal impetus has been biological; for example, the results bear on the formation of morphogenetic fields in developing organisms. The models of Gierer and Meinhardt have  $n = 2$ , the two reacting components being an "activator"  $u$  and an "inhibitor"  $v$ . These authors give arguments to explain why patterns may be expected, if the inhibitor diffuses more rapidly than the activator. (There is some analogy with Segel and Jackson's arguments involving stabilizers and destabilizers).

Here we deal with the following "inverse" problem: given a pair of functions  $(\varphi(x), \psi(x))$ , find a reaction-diffusion system with  $n = 2$ , for which  $(\varphi, \psi)$  is a pattern. Note that our theorem in Sec. II indicates that the corresponding problem with  $n = 1$  does not have a solution, unless  $\varphi$  is monotone; and even then, the pattern is structurally unstable. We prove that for "single peak" distributions  $(\varphi, \psi)$  of a certain type, in which  $\varphi$  and  $\psi$  are related linearly, a solution of the inverse problem does exist (in fact, many solutions exist). We also show that for "multiple peak" distributions, solutions

exist if the "stability" requirement for patterns is weakened to the statement that the spectrum of the linearized operator has no points in the unstable half-plane. It should be emphasized that it is easy to find systems for which a given distribution  $(\phi, \psi)$  is a solution; but we must also prove that  $(\phi, \psi)$  is stable for the system, a considerably more difficult task. The systems we construct are of the activator-inhibitor type, and the inhibitor does (as in [14], [19], [20]) diffuse more rapidly than the activator. Our results are not likely to be of practical importance, because model systems (1) which have been used in the past, and probably future models as well, have other desired properties besides merely being of activator-inhibitor type. Thus, the source functions  $f$  and  $g$  used in [14], [19], [20] were built on the basis of more or less specific morphogen reactions. At the same time, in constructing model reaction-diffusion systems with only two components, one should not attach overriding importance to having them mirror specific reaction networks involving the two species. In fact, the actual mechanisms modelled will involve a large number of reacting species; if one pictures the reduction to two as having been made through various pseudo-steady-state, slow reaction, or other approximations, then the connection between the source terms  $f$  and  $g$  and the actual kinetics will necessarily be obscured. In any case, the present analysis does shed light on the role which activation and inhibition mechanisms, with distinct migration rates, have in inducing stability of nonuniform structures.

We deal with a two-component system

$$(20a) \quad u_t = u_{xx} + f(u, v)$$

$$(20b) \quad v_t = kv_{xx} + g(u, v).$$

It has been nondimensionalized so that the diffusion coefficient of  $u$  is 1; we let  $k$  be that of  $v$ .

Definition:  $u$  is an activator if  $f_u > 0$ ,  $g_u > 0$ , for the ranges of  $u$  and  $v$  at hand.  $v$  is an inhibitor if  $f_v < 0$ ,  $g_v < 0$  for this range.

Thus, increasing the amount of the activator increases the rate of production of  $u$  and  $v$ , whereas increasing  $v$  has the opposite effect.

First, we deal with "single peak distributions". Let  $\varphi(x), \psi(x)$  be a pair of  $C^5$  functions which, as in Fig. 5, are even in  $x$ , approach limits as  $|x| \rightarrow \infty$ , satisfy

$$\varphi'(x) \neq 0 \text{ for } x \neq 0, \varphi''(0) \neq 0, \varphi = h(\psi)$$

for some function  $h$  with  $h'(\psi) > 0$  in the closure of the range of  $\psi$ , and such that  $\frac{\varphi''(x)}{\varphi(x) - \varphi(\infty)}$  approaches a nonzero finite limit as  $|x| \rightarrow \infty$ .

Theorem 1: Let  $(\varphi, \psi)$  satisfy the above assumptions. Let  $k > 1$ . Then there exist functions  $f(u, v)$ ,  $g(u, v)$ , and  $p(v)$  such that  $f_u > 0$ ,  $g_u > 0$ ,  $f_v < 0$ ,  $g_v < 0$ ,  $p > 0$ , and  $(\varphi, \psi)$  is a  $C^0$ -stable stationary solution of

$$(21a) \quad u_t = u_{xx} + f(u, v)$$

$$(21b) \quad v_t = k(p(v))^{-1} (p(v)v_x)_x + g(u, v).$$

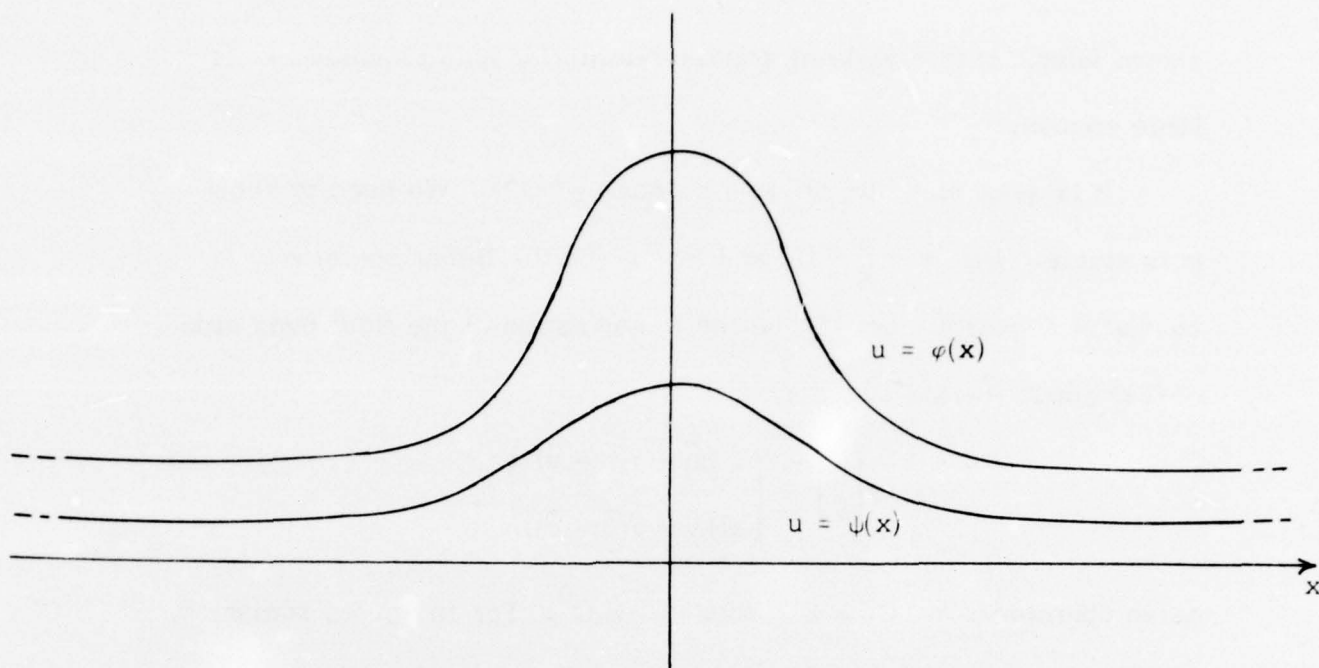


Fig. 5. Typical single-peak distribution.

Moreover if  $\varphi = c_1\psi + c_2$  for some constants  $c_i$  with  $c_1 > 0$ , then  $p \equiv 1$ , so that our system is of the form (20).

Remark: In fact, our construction will yield many solutions of the inverse problem.

Sketch of proof: First we consider the case  $\varphi \equiv \psi$ . Under the assumptions given, there is a  $C^2$  function  $F(u)$  such that  $\varphi'' = -F(\varphi)$ . The simplest system for which  $(\varphi, \psi)$  is a pattern will be of the form

$$(22a) \quad u_t = u_{xx} + F(u) + \sigma(u - v)$$

$$(22b) \quad v_t = kv_{xx} + kF(v) + k\sigma(u - v),$$

where  $\sigma$  is a sufficiently large real number. Other examples will be



shown later. It is clearly of activator-inhibitor type provided  $\sigma$  is large enough.

It is seen that  $(\varphi, \psi)$  is a solution of (22). We need to show it is stable. Let  $\varepsilon = \frac{1}{k} < 1$ , and let  $L$  be the linear operator  $Lu \equiv u'' + F'(\varphi(x))u$ . Let  $S$  be the linearization of the right hand side of (22) about  $(\varphi, \psi)$ :

$$S \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} L\bar{u} + \sigma(\bar{u} - \bar{v}) \\ k(L\bar{v} + \sigma(\bar{u} - \bar{v})) \end{pmatrix},$$

as an operator from  $C^2 \times C^2$  into  $C^0 \times C^0$ . For any given scalar  $\lambda$ , let  $P_\lambda(L)$  be the operator

$$P_\lambda(L) \equiv \varepsilon \lambda^2 + \lambda[(1 - \varepsilon)\sigma - (1 + \varepsilon)L] + L^2,$$

acting on functions in  $C^4(\mathbb{R})$ . If  $\lambda$  is such that  $P_\lambda(L)$  is boundedly invertible on  $C^0(\mathbb{R})$ , then it turns out that  $(S - \lambda)$  is boundedly invertible on  $C^0(\mathbb{R}) \times C^0(\mathbb{R})$ , so that  $\lambda \notin \Sigma(S)$  (spectrum of  $S$ ). This is proved by actually writing down a solution of  $(S - \lambda) \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$  in terms of  $P_\lambda(L)^{-1}$  and  $L$ . Hence  $\Sigma(S) \subset \{\lambda : \Sigma(P_\lambda(L)) \ni 0\}$ , so that information about the spectrum of  $S$  can be found from knowledge of  $\Sigma(P_\lambda(L)) = P_\lambda(\Sigma(L))$ .

The following facts about the function  $P_\lambda(\mu)$  can be verified directly: For fixed  $\mu$ , it is a quadratic polynomial in  $\lambda$ . Given any  $M$ , its two roots will be real and negative for any  $\mu < M$  and  $\mu \neq 0$ , as long as  $0 < \varepsilon < 1$  and  $(1 - \varepsilon)\sigma$  is large enough (depending on  $M$ ).

(Incidentally, this is where the condition  $k > 1$  is seen to be necessary.) If  $\mu = 0$ , however, one of the roots will be zero and the other negative. Now the spectrum  $\Sigma(L)$  is, in fact, real, bounded from above by some number  $M$ , and contains  $0$  as a simple isolated eigenvalue. This follows from our assumptions on  $\varphi$ , and from known facts about the spectra of second order operators. Thus as  $\mu$  ranges over  $\Sigma(L)$ , the set of roots  $\lambda$  of  $P_\lambda(\mu)$  will range over a set of nonpositive real numbers which contains the origin once, and is otherwise bounded away from zero. In view of the fact (shown in the previous paragraph) that  $\Sigma(S) \subset \Lambda$ , we know that  $\Sigma(S)$  has the same property.

But with this knowledge about the spectrum of  $S$ , we can apply Sattinger's stability theorem [24] to deduce the  $C^0$ -stability of the solution  $u = \varphi$ ,  $v = \varphi$ . (Incidentally, I believe this to be one of the first examples in which Sattinger's theory has been applied to specific systems of order greater than one.)

Now consider the case when  $\varphi = h(\psi) \not\equiv \psi$ . We write the system (22) with  $v$  replaced by the symbol  $w$ , then transform the second dependent variable according to the equation  $w = h(v)$ , to obtain a system of the form (21), with  $p(v) = h'(v)$ . Clearly if  $h' = \text{const}$ , we may take  $p \equiv 1$ . This completes the proof.

Next, we consider distributions with many peaks, in fact those which are periodic in  $x$ . Let  $\varphi(x)$  and  $\psi(x)$  be periodic  $C^5$  functions such that  $\varphi'' = -F(\varphi)$  for some  $C^2$  function  $F$ , and  $\varphi = c_1\psi + c_2$ ,  $c_1 > 0$ .

Theorem 2: Let  $\varphi$  and  $\psi$  satisfy these assumptions. Let  $k > 1$ .

Then there exist functions  $f, g$  satisfying the inequalities stated in Theorem 1, such that  $(\varphi, \psi)$  is a solution of (20). Moreover, the spectrum of  $S$  (the linearization of the right hand side of (20) about  $u = \varphi, v = \psi$ ) is contained on the nonpositive real axis.

Remark: The fact that none of the spectrum of  $S$  is in the unstable half-plane is an indication of the stabilizing influence of the activator-inhibitor mechanism, although it is not proved that  $(\varphi, \psi)$  is stable in the  $C^0$  sense.

The proof of Theorem 2 proceeds as did the other theorem, with the difference that it is no longer true that 0 will be an isolated point of the spectrum of  $L$ .

Other solutions of the inverse problem: Our solutions of the inverse problem (in the case  $\varphi = \psi$ ) have been of the form (22). Equally valid solutions can be obtained by replacing the constant  $\sigma$  in the second equation (22b) by a different large constant  $\tau$  in a certain range depending on  $\sigma$ , and by adjoining extra terms  $(u - v)^2 \tilde{f}(u, v)$  and  $(u - v)^2 \tilde{g}(u, v)$  to the right sides of (22a) and (22b), respectively, where  $\tilde{f}$  and  $\tilde{g}$  are arbitrary. The given distributions will of course still be solutions of the revised equations, and the stability analysis is unchanged. Moreover, the inequalities given in the statement of the theorem hold true for  $u$  near  $\varphi, v$  near  $\psi$ , but are no longer necessarily true for arbitrary  $u$  and  $v$ .

Example: The function  $\varphi(x) = \frac{6e^{x/d}}{(1 + e^{x/d})^2}$  is an even peak-like

distribution with  $\varphi(0) = 3/2$ ,  $\varphi(\infty) = 0$ , and with peak-width of the

order  $d$ . It can be checked that  $\varphi$  satisfies the equation  $\varphi'' + d^{-2}(\varphi^2 - \varphi) = 0$ .

Setting  $\psi(x) = a\varphi(x)$ ,  $a > 0$ , we see that  $(\varphi, \psi)$  will be an activator-inhibitor pattern for the system

$$u_t = u_{xx} + d^{-2}(u^2 - u) + \sigma(u - av) + (u - av)^2 \tilde{f}(u, v),$$

$$v_t = kv_{xx} + kd^{-2}(av^2 - v) + \sigma(u - av) + (u - av)^2 \tilde{g}(u, v),$$

provided that  $k > 1$ , and  $\sigma$  is sufficiently large. The functions  $\tilde{f}$  and  $\tilde{g}$  are arbitrary.



### III. B. 2. Plateaus

The previous section is relevant when one wishes to model the concentration of reacting substances at isolated locations. The present section, on the other hand, is concerned with modelling phenomena of differentiation, characterized by the development of sharp boundaries between regions within which the concentrations are relatively uniform. We call these sharp boundaries "transition layers" (see [9]).

The methods of singular perturbations may be used to construct patterns with sharp transition layers. This was explained in [10] for boundary value problems on finite domains in higher dimensional space, and an example was given in [9] of transition layer solutions in one independent variable, though no discussion of stability was given in the latter paper.

Here we review the construction process for periodic plateau-type patterns on the whole line. In this case, contrary to that studied in [10], there will be no boundary effect on the appearance of our dissipative structures, because there is no boundary. The following will be formal and nonrigorous.

As in [10], we take as model system the following:

$$(23a) \quad u_t = \epsilon^2 u_{xx} + f(u, v)$$

$$(23b) \quad v_t = v_{xx} + g(u, v),$$

where  $\epsilon$  is small, and where the source term  $f$  has the characteristics shown in Fig. 6 (see [11] for references to reaction networks realizing this type of source function).

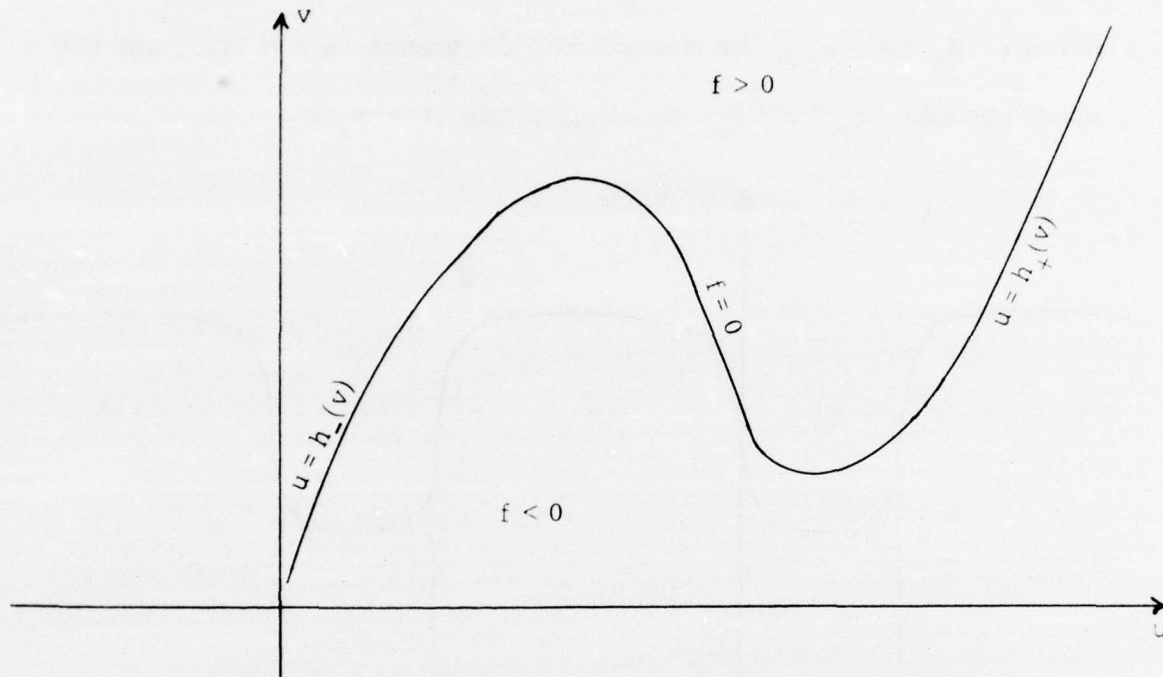


Fig. 6. Regions of positivity and negativity for the source function  $f(u, v)$ .

Our object will be to construct periodic stationary solutions having the appearance shown in Fig. 7. As seen there, the upward transitions occur roughly at locations  $\alpha_n$  with  $\alpha_n = \alpha_0 + np$ , where  $p$  is the wave-length; and the downward transitions are at  $\beta_n = \beta_0 + np$ . On the relatively "flat" portions between the transitions, (23a) with  $u_t = \epsilon = 0$

should be satisfied approximately. That is,  $f(u, v) = 0$ , so the phase plane image should lie near the S-shaped curve in Fig. 6. Referring to that figure, where the two ascending branches  $u = h_{\pm}(v)$  are attractors for the corresponding kinetic equations, we specify that the lower flat regions  $\beta_n < x < \alpha_{n+1}$  be mapped onto the branch  $u = h_-(v)$ , and the upper portions  $\alpha_n < x < \beta_n$  be mapped onto  $u = h_+(v)$ .

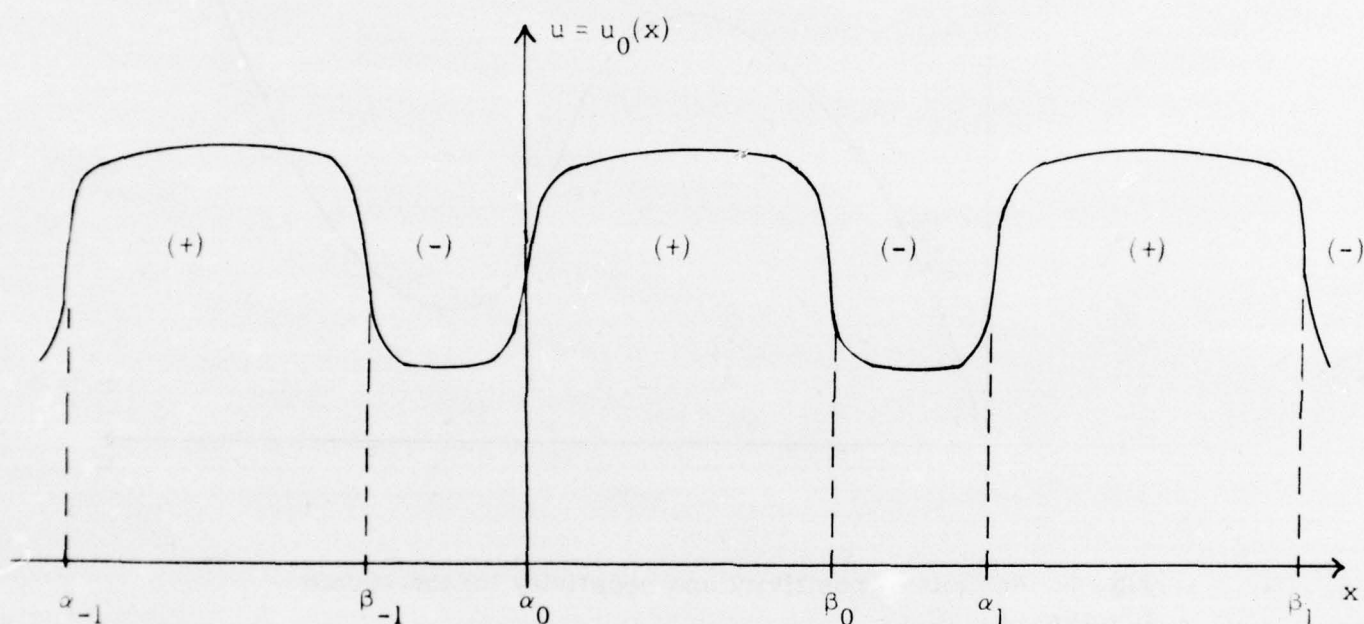


Fig. 7. Plateau-type pattern for the  $u$ -component.

We are seeking solutions for which  $v$  does not change abruptly through the transition layer, although  $u$  does. Therefore to lowest order, there is a well-defined value of  $v$  at each transition point. It is explained in [10] that the only value of  $v$  which can support a transition layer is a value  $v^*$  for which

$$(24) \quad J(v^*) = 0 ,$$

where  $J(v) \equiv \int_{h_-(v)}^{h_+(v)} f(u, v) du$ . Let us assume there is only one value

of  $v$  satisfying (24). The explanation of (24), as well as the construction of the solution inside the layer itself, is along the following lines.

Suppose the transition occurs at  $x = 0$ , where  $v = v_0$ . We stretch variables by setting  $\xi = x/\epsilon$ , then (23a) with  $u_t = 0$  becomes, to lowest order in the new variables,

$$(25) \quad \frac{d^2 u}{d\xi^2} + f(u, v_0) = 0 .$$

The methods of matched asymptotics dictate that (25) should be solved for  $\xi \in (-\infty, \infty)$  with boundary conditions (if it is an upward transition)  $u(-\infty) = h_-(v_0)$ ,  $u(+\infty) = h_+(v_0)$ . Assuming a solution  $u(\xi)$  exists, multiply (25) by  $\frac{du}{d\xi}$  and integrate the result from  $-\infty$  to  $+\infty$  to obtain

$$0 = \int_{-\infty}^{\infty} \left[ \frac{1}{2} \frac{d}{d\xi} \left( \left( \frac{du}{d\xi} \right)^2 \right) + f(u, v_0) \frac{du}{d\xi} \right] d\xi = \int_{h_-(v_0)}^{h_+(v_0)} f(u, v_0) du = J(v_0) ;$$

hence  $v_0 = v^*$ . It turns out that this is a sufficient, as well as a necessary, condition for the existence of a solution of (25) with the given boundary conditions.

At this point, we have obtained the following information. On the broad tops and bases of the plateau configurations in Fig. 7,  $u$  and  $v$  are related by  $u = h_{\pm}(v)$ . At the transition points between them,  $v = v^*$ .



We now proceed to a determination of  $v(x)$ . In (23b), we replace the variable  $u$  by  $h_{\pm}(v)$ , depending on whether we are on the top or the base of a plateau. We thus ignore the fine structure of the transition, replacing it by an actual discontinuity. Then (23b) (with  $v_t = 0$ ) becomes

$$(26) \quad v_{xx} + G(v, x) = 0,$$

where

$$G(v, x) = \begin{cases} g_-(v) \equiv g(h_-(v), v) & \text{for } \beta_n < x < \alpha_{n+1}, \\ g_+(v) \equiv g(h_+(v), v) & \text{for } \alpha_n < x < \beta_n. \end{cases}$$

This equation, together with the condition  $v = v^*$  for  $x = \alpha_n$  or  $\beta_n$ , can be expected typically either to have no solution, or a one-parameter family of periodic solutions (translations in  $x$  would give a second parameter, which we ignore). For example, suppose  $g_-(v) < 0$ . Then  $v_{xx} > 0$  for  $x \in (\beta_n, \alpha_{n+1})$ , which implies  $v < v^*$  for  $x \in (\beta_n, \alpha_{n+1})$  and  $v_x(\beta_n) < 0$ ,  $v_x(\alpha_n) > 0$ . If  $g_+(v) < 0$  as well, then the same derivative conditions must hold at the endpoints of the intervals  $(\alpha_n, \beta_n)$ . But these two sets of derivative conditions contradict one another, so there can be no solution.

For a pattern to exist, it is therefore necessary that  $g_{\pm}(v)$  not be of one sign. As a simple example, let us assume that  $g_{\pm}(v)$  are constants with  $g_- < 0 < g_+$ . Then from (26), we have  $v'' = -g_+$  for  $x \in (\alpha_n, \beta_n)$ ; hence setting  $p_+ = \beta_n - \alpha_n$ , we get

$$v = v^* + \frac{1}{2} g_+ (x - \alpha_n)(\beta_n - x), \quad x \in (\alpha_n, \beta_n); \quad v'_n(\beta_n) = -\frac{1}{2} g_+ p_+.$$

Similarly,  $v = v^* + \frac{1}{2} g_- (x - \beta_n)(\alpha_{n+1} - x)$  in  $(\beta_n, \alpha_{n+1})$ ,  $v'_n(\beta_n) = \frac{1}{2} g_- p_-$ , where  $p_- = \alpha_{n+1} - \beta_n$  (so that  $p_- + p_+ = p$ ). Matching the derivatives at  $\alpha_n$  yields the condition

$$(27) \quad g_+ p_+ = -g_- p_-.$$

Thus in this simple example, we apparently obtain a plateau-like solution for any spacing  $p_{\pm}$  of the  $\alpha$ 's and  $\beta$ 's satisfying (27). However, this is only partly true: the wavelength  $p$  cannot be too large, for then the range of the function  $v$  might extend beyond the domains of one or both of the functions  $h_{\pm}$ .

The foregoing describes a method of obtaining formally approximate solutions for small  $\varepsilon$ . The procedure could be continued to yield higher order approximations; the constructions in [10] and [11] are relevant here.

Now supposing a plateau solution (which we denote by  $(u_0(x), v_0(x))$ ) has been constructed, how can we infer its stability or instability? The following heuristic argument suggests a workable criterion.

Suppose we subject the given stationary solution to a perturbation which is small in the  $C^0$  norm. Such a perturbation, of course, does not erase or relocate the transition layers. For the moment, let us assume that the latter do not, in fact, move during the subsequent evolution of the perturbed system. Then between the layers, the image points  $(u, v)$  will again be drawn to the attractors  $h_{\pm}(v)$  (by neglecting  $\varepsilon$  in (23a)), and at the same time, the evolution of  $v$  will be governed

by (23b). Eventually, the term  $g$  in this latter equation may be replaced by  $G(v, x)$  (see (26)):

$$(28) \quad v_t = v_{xx} + G(v, x) .$$

The solution before perturbation was such that  $v_0$  satisfied (28) and was time-independent, of course. So we must now investigate the stability of solutions of (28). If we impose the condition

$$(29) \quad \frac{d}{dv} g_{\pm}(v) < 0$$

(this being the  $v$ -derivative along the ascending branches in Fig. 6), then the solution  $v_0(x)$  will be stable. In fact, for small positive constants  $\delta$ ,  $v_0(x) \pm \delta$  will be super- and sub-solutions of (28), so again as in [1], solutions of (28) with initial data caught between  $v_0 - \delta$  and  $v_0 + \delta$  will converge uniformly to  $v_0$ .

This argument asserts stability under condition (29) if we assume the transition layers are immobile. They are not, in fact, immobile, but their movement is slow and can be analyzed by the techniques in [11]. Immediately following the initial perturbation, the value of  $v$  at any given transition layer will no longer necessarily satisfy  $J(v) = 0$  (as in (24)). What this means is that the layer will develop into a sharp wave front which, locally, will be of the form  $u = U(\frac{x - \epsilon ct}{\epsilon}, x)$ , where  $c$  depends on  $v$  at the position of the front, so will itself change with time. This approximate representation of  $u$  follows the argument in [10], [11], to which the reader is referred for more details. As stated

above, the velocity  $\epsilon c$  will be small, so that until a time of the order  $1/\epsilon$  has elapsed, the above assumption that the layer is immobile should be valid.

But after this time, shifts in the transition layer's position due to the slow motion of the front may have accrued.

The effect of changes in the layer's position should therefore be determined. It turns out that the nature of this effect depends on the signs of the two functions  $g_{\pm}(v)$ . In the following, we call the collection of intervals on which  $u \sim h_{+}(v)$  the domain of the (+) state, and similarly for the (-) state. Suppose, for example, that  $g_{-}(v) < 0 < g_{+}(v)$ . It will then also be true that  $g(u, v) < 0$  when the point  $(u, v)$  is in a neighborhood of the left ascending branch in Fig. 6, and  $g > 0$  near the right ascending branch. We focus attention on the effect of shifts in the position of a single transition layer. Suppose, initially, that we are in a stationary state, and that the points  $(u, v)$  lie in the two neighborhoods described above on opposite sides of the layer. Now suppose the position of the layer is perturbed in the direction which will increase the "domain" of the state (+) (that is to say, it is moved away from the (+) state toward the (-) state. Since  $g_{+} > g_{-}$ , the immediate effect of this is to increase the source term  $g(u, v)$  in (23b), in the interval between the old and the new positions. This will cause the function  $v_t(x, t)$ , hence  $v$ , to increase. At the same time, the value of  $v$  at the layer has also changed by virtue of the fact that the initial stationary function  $v(x)$



was not constant, whereas the layer's position was altered. This second effect may, for a period of time, counteract the first effect (the increase of  $v$  due to increasing the source terms); but eventually the first will dominate, because the subsequent motion of the layer is slow, and  $v_t$  will remain positive until the position has moved a significant amount. So eventually the value of  $v$  at the layer will surpass its initial value.

Let us now assume that  $J(v)$  is an increasing function of  $v$  (apparently the case for the function  $f$  depicted in Fig. 6). Then such an increase in the value of  $v$  at the front also raises  $J$ , which in turn results in the (+) state becoming more dominant, in the terminology of [10] and [11]. This means that the (slow) motion of the front will proceed in a direction which further increases the domain of the (+) state. The perturbed layer will therefore not tend to return to its original location, and we are in an unstable situation.

On the other hand, if we assume the opposite inequalities

$$(30) \quad g_+(v) < 0 < g_-(v),$$

then the above analysis indicates that a perturbed position of the layer will eventually move back toward its original location. Essentially the same analysis holds if the perturbation is in the opposite direction.

On the basis of the above arguments, which admittedly are far from being rigorous, I conjecture that the constructed periodic plateau-type stationary solution will be stable if (29) and (30) hold, and if  $J$  is an increasing function of  $v$ .

In [10], an argument for stability or "realizability" was given, in some respects similar to the above. Yet in other respects, it was different, because the effects of boundary conditions in a bounded domain were essentially involved.

#### IV. Discussion

This paper is devoted entirely to patterns on the entire real line, and it may well be objected that real reacting and diffusing systems are always finite. Suppose the real system is one-dimensional, and its domain is a finite interval  $I$ . If the length of  $I$  is large compared with the characteristic length of the patterns formed, then the infinite-line approximation should be a good one, in the interior of  $I$ . In fact, the patterns studied here are generated by the processes of reaction and diffusion only, and so should exist in interior regions independently of any boundary effects. Of course there also exist patterns which are generated or strongly conditioned by the presence of boundaries and by the transport mechanisms occurring there. This type of pattern is not subject of the paper. In short, our restriction to the infinite line was made for the purpose of exploring pattern phenomena generated solely by reaction and diffusion, without having to worry about boundary effects.

Nevertheless, some things can be said immediately about certain patterns on finite domains, analogous to those discussed here.

Consider first the scalar case,  $n = 1$ . This is a matter of solving (2) on an interval, with given boundary conditions. We restrict attention to homogeneous Neumann (no-flux) or to Dirichlet ( $u = 0$ ) boundary conditions. Again, all solutions can be constructed, and their stability analyzed by techniques similar to those described. It turns out that under Neumann conditions, the only stable solutions are certain constant ones, and under Dirichlet conditions, the only stable ones are those which have



a single internal extremum (except in the case when  $u \equiv 0$  is stable). See also the treatment in [ 5 ].

Consider next the question of small amplitude patterns, as treated in III. A. Set on a finite interval with homogeneous boundary conditions, this is a standard bifurcation problem, and the results of [ 6 ], [ 7 ], [ 23 ] apply. One commonly finds stable super-critical bifurcation branches of nonconstant solutions which, for small amplitudes, approximate periodic functions with many wavelengths fitted into the given interval ([ 2 ], [ 4 ]). These will also approximate some of the periodic solutions found in III. A.

The treatment of peaks--single or multiple-- in III. B. 1. may be extended with little change to the case of a finite interval with (say) Neumann boundary conditions imposed. (Of course, one must begin with distributions  $\varphi$  and  $\psi$  which, themselves, satisfy these boundary conditions). The same is true of the plateaus in III. B. 2. See especially [ 9 ] and [ 10 ].

A second objection may be that real reacting and diffusing systems are distributed in space or on a surface, rather than on a line, as we have here. It is probable that the extension of the present results to problems on  $\tilde{R}^2$  or  $\tilde{R}^3$  will entail many complications and difficulties. In the case  $n = 1$ , for example, one can no longer immediately write down all stationary solutions of  $\Delta u + f(u) = 0$ ; and though in principle the technique of Aronson and Weinberger works in space, its application may be difficult.



As for small amplitude patterns in the case  $n > 1$ , this problem is fraught with most of the difficulties associated with the Bénard problem in fluid mechanics. One can find solutions which are doubly periodic with various kinds of symmetries, but the stability question is very difficult, unless one allows only perturbations within certain classes restricted by artificial symmetry conditions. Of course, if one wants to find patterns for which boundary effects are important, one has a boundary value problem in a bounded domain. Typically, this reduces to a standard bifurcation problem, and the identification of the bifurcation points, together with the determination of the approximate shape of the small amplitude patterns which arise, involves solving a linear eigenvalue problem.

There are real complications involved in extending the treatment in III. B. 1 to higher dimensions. On the other hand, plateau configurations (as in III. B. 2) can be constructed, and was done in [10] (see also [12]).

A final comment should be made about the terminology "pattern formation". The same term is also commonly used in developmental biology, where it refers to structural organization in developing organisms. The possible relevance of reaction-diffusion patterns, such as those in the present paper, to patterns in this other sense, will suggest itself. No claim is being made here that reaction-diffusion patterns are anywhere near to being adequate models for the biological patterns. In fact, the

latter are undoubtedly highly complicated phenomena involving many other types of processes besides those suggested here. At the same time, any serious student of theoretical developmental biology should be aware of the rich repertoire of solutions, including stable patterned ones, exhibited by rather simple models based on reaction and diffusion alone. To this extent, I hope the results given in this paper may be useful to the biologist.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1709	2. GOVT ACCESSION NO. (9)	3. RECIPIENT'S CATALOG NUMBER Technical
4. TITLE (and Subtitle) STATIONARY PATTERNS FOR REACTION-DIFFUSION EQUATIONS.		5. TYPE OF REPORT & PERIOD COVERED Summary Report, no specific reporting period
7. AUTHOR(s) 10 Paul C. Fife		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		8. CONTRACT OR GRANT NUMBER(s) 15 DAAG29-75-C-0024 ✓ NSF - MPS-74-06835
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 53p.		12. REPORT DATE 11 January 1977
		13. NUMBER OF PAGES 50
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) 14 MRC-TSR-1709 Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709 National Science Foundation Washington, D. C. 20550		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Patterns stability reaction-diffusion systems bifurcation of periodic solutions semilinear parabolic systems		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Patterns are defined to be stable stationary nonconstant solutions of the equations of reaction and diffusion. Several approaches are used to show the existence (or nonexistence) of patterns depending on one variable and defined on the entire real line. For a scalar equation, it is shown that there are essentially no patterns. For a system, small amplitude patterns, larger amplitude "peaks", and larger amplitude "plateaus" are treated. In all cases, stability is an important consideration. Applications to ecology and biophysics are mentioned.		

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